Primal-Dual Algorithms for Weighted Abstract Path and Cut Packing

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Outline

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   - Integral LPs

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2. Hoffman's Models
   - Packing problems
   - Path models
   - Cut models
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4. Extensions
   - Flows over Time
   - Parametric Capacities

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- Here we proceed in this same spirit.

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- These formulations do *not* in general work for weighted versions.
  - E.g., if we put general "rewards" on paths, then Max Weighted Path Flow is NP Hard.

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Question 3: Can we find polynomial algorithms for these abstract weighted path and cut packing problems?

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- This generic problem has many applications, e.g., flow is packing paths into arcs, connectivity is packing trees into edges, etc.

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$$\text{(D)} \quad \max \sum_{D} r_D y_D \qquad \text{s.t.} \sum_{D \ni e} y_D \le u_e \quad \forall e \in E \qquad y \ge 0$$

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Big Question: When do these LPs have guaranteed integer optimal solutions?

An example packing LP

- Consider:

$$\max \mathbb{1}^T y$$

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- What if we change the RHS u? The objective r?

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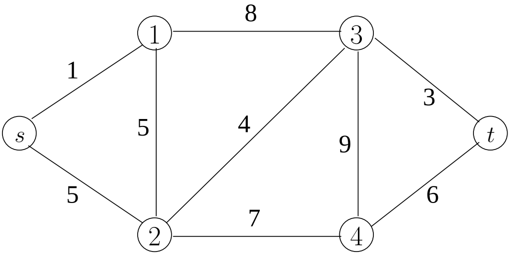
- How do I know that the first two objectives are "good" for all RHS?

How the example was constructed

- Consider the following graph:



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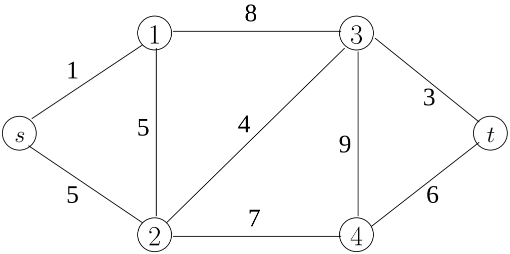

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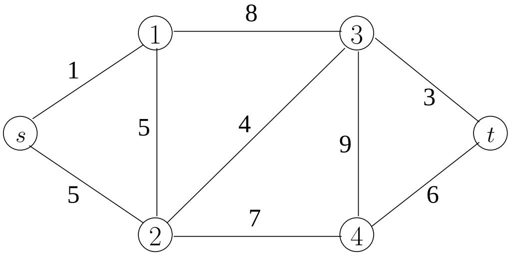

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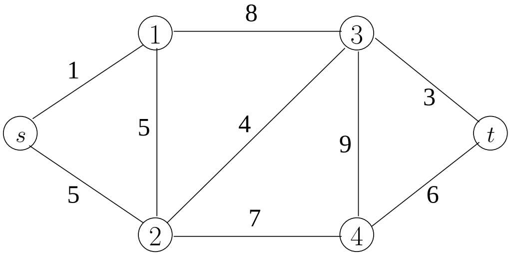

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- Why does this lead to integer optimal LP solutions?

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- And in fact Dijkstra's Algorithm gives an integer optimal solution to this form of Shortest Path.

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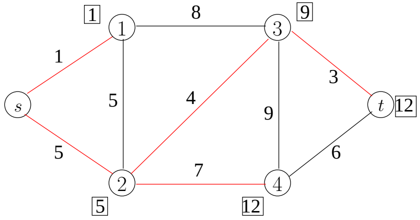
- Here is the Dijkstra solution with its shortest path tree:



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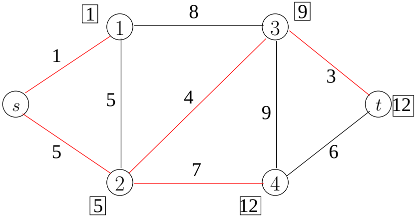
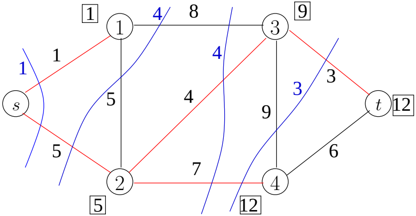


- Recall that we can greedily construct a tight cut packing that proves that this shortest path tree is optimal:



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- Hoffman did it . . .

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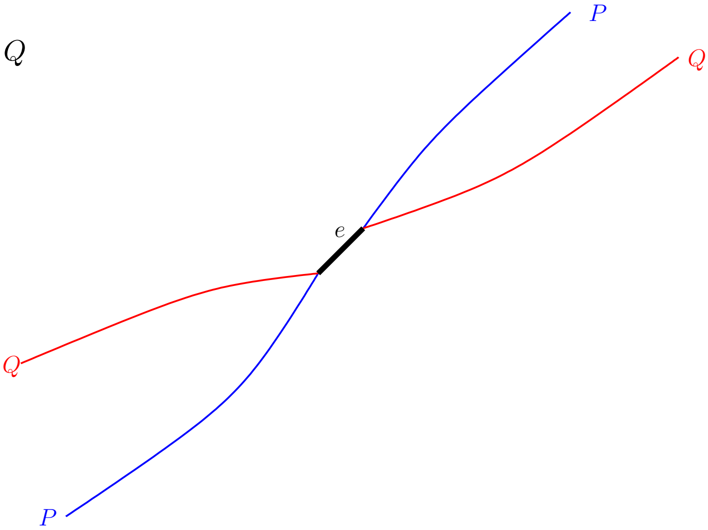
Picture of Crossing Axiom

$e \in P \cap Q$



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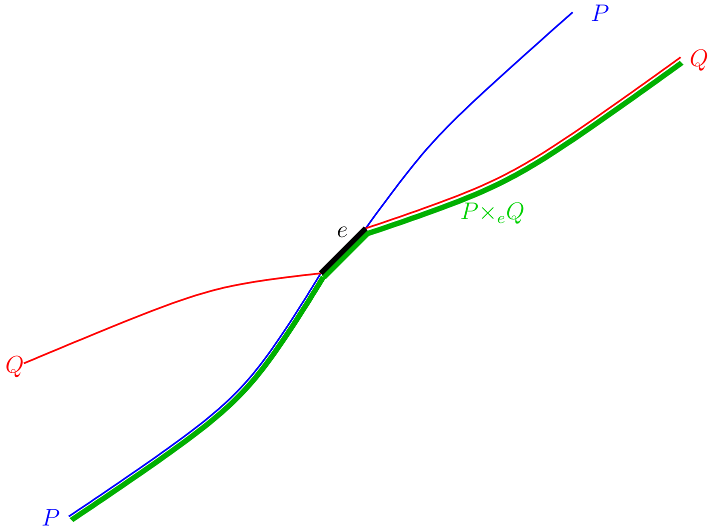

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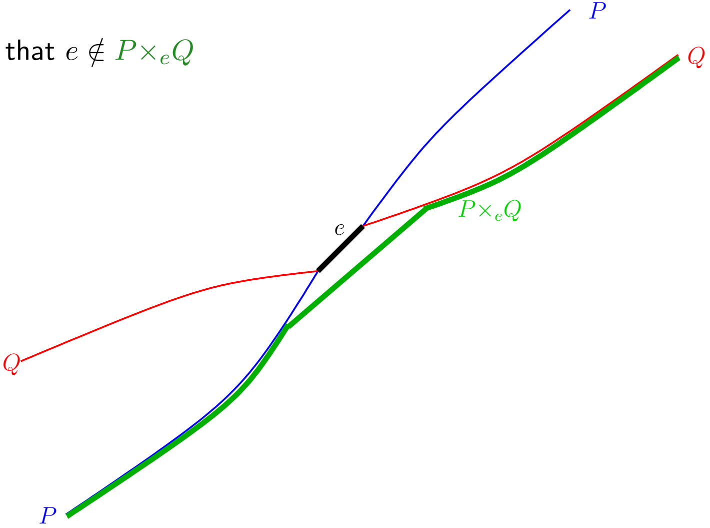

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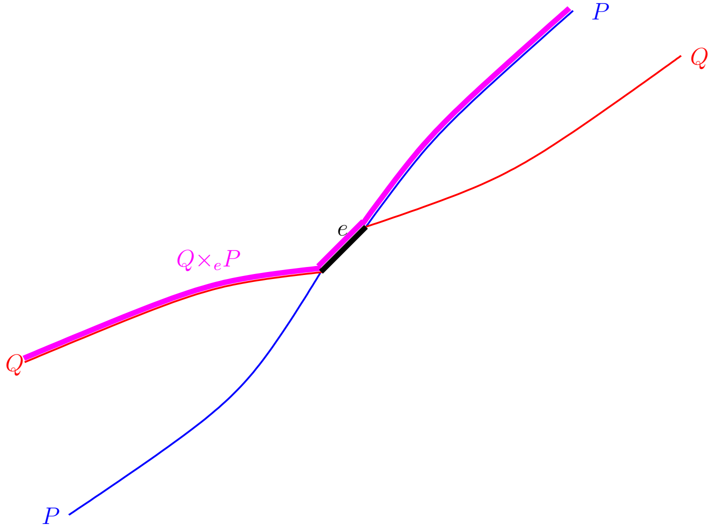

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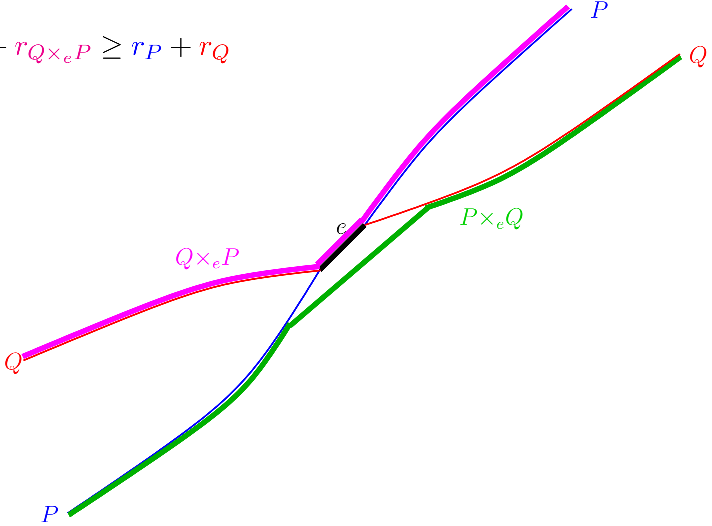

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  - Alan earlier verbally told me that he put in the supermodular r because he wanted to imitate the nice things that Jack Edmonds was doing.

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- At lunch afterwards Bill Pulleyblank accosted some of us and said something like "surely some of you young guys should be able to answer Alan's question".
- As a bonus, Bill relayed to us Alan's concrete suggestion for an oracle for the max flow ($r \equiv 1$) version: You send the oracle a subset S of the elements, and it tells you whether there is a path P with $P \subseteq S$ (and $<_P$) or not.

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Understanding the Cut Axioms

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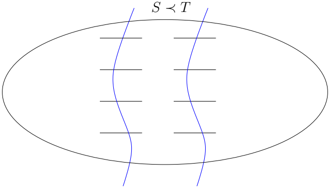

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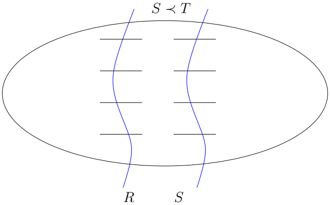

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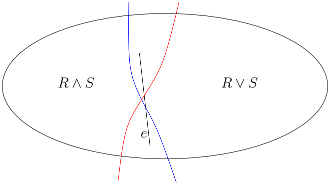

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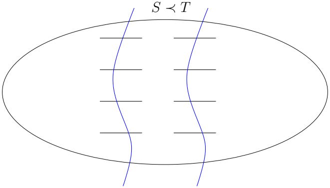

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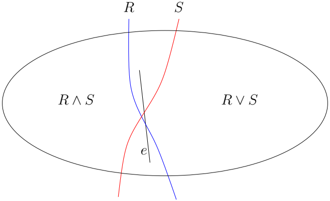

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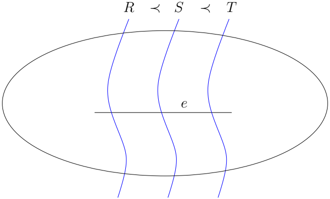

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Other applications

Lattice polyhedra would not be so interesting unless they included interesting applications other than Shortest Path:

- Dilworth's Theorem (chains and antichains in posets) and various Greene-Kleitman generalizations.
- Shortest Path in hypergraphs.
- Polymatroids and intersections of polymatroids.
- Min-cost arborescence.
- Our example with $r = (4\ 3\ 2\ 3\ 1\ 1\ 3\ 2\ 4)$ has integer optimal solutions for all RHS u because this r is supermodular: each $r_D = 6 - \#$ edges crossing D.
- Our example with $r = (0\ 9\ 0\ 0\ 9\ 0\ 0\ 9\ 0)$ can have a fractional solution because this r is not supermodular.
- Etc, etc ...

Blocking Carries Over

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What remains now is Q3:

Are there polynomial algorithms for solving Weighted Abstract Flow and Cut Packing?

Outline

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- Each iteration maintains that x and π are optimal for current flow value, so when x becomes a max flow, it is optimal.

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- But otherwise, the advantage of P-D is that it replaces the complicated objective $r^T x$ with a simple objective $\mathbb{1}^T x$.
- Due to R, the solution to the restricted dual could have -1 values in it, so the dual update need not be monotone.

P-D for Path and Cut Packing 1

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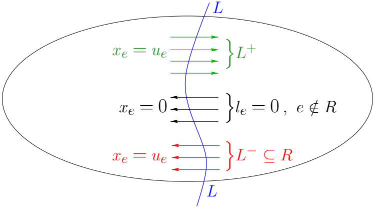

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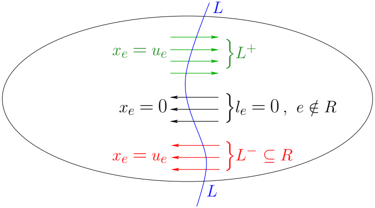

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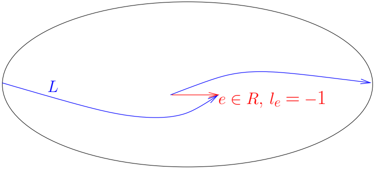

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P-D for Path and Cut Packing 5

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- Theorem: This algorithm solves Weighted Abstract Flow in weakly polynomial time.

Cut Packing

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Outline

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- Same idea works for abstract networks, but need to repeat path flows over time.

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- Thus we can solve max abstract flow over time in polynomial time (modulo lots of details).

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- This application to max abstract flow finally gives us an application where the supermodularity was really necessary.

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    - Extended by Gusfield and Martel; Mc; F. Granot, Mc, Queyranne, Tardella; . . .

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- Then parametric abstract flows over time :-) ?

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5. Could we further extend this idea to solve, e.g., Schrijver's general framework for TDI problems?
6. One can make a good career out of answering open questions in Alan's papers :-)

Dedication

I dedicate this talk to
Alan Hoffman's 90th birthday,
and to his long and fruitful career.

Questions?

Comments?