Colouring graphs with no odd holes

Paul Seymour (Princeton)
joint with Alex Scott (Oxford)

Chromatic number $\chi(G)$: minimum number of colours needed to colour G.

Chromatic number $\chi(G)$: minimum number of colours needed to colour G.

Clique number $\omega(G)$: size of largest clique in G.

Theorem (Tutte, 1948)

There are graphs G with $\omega(G) = 2$ *and* $\chi(G)$ *arbitrarily large.*

Theorem (Tutte, 1948)

There are graphs G with $\omega(G) = 2$ and $\chi(G)$ arbitrarily large.

Hole: induced subgraph of G which is a cycle of length > 3.

Theorem (Tutte, 1948)

There are graphs G with $\omega(G) = 2$ and $\chi(G)$ arbitrarily large.

Hole: induced subgraph of G which is a cycle of length > 3.
Antihole: induced subgraph of G which is the complement of a cycle of length > 3.

Theorem (Tutte, 1948)

There are graphs G with $\omega(G) = 2$ and $\chi(G)$ arbitrarily large.

Hole: induced subgraph of G which is a cycle of length > 3.
Antihole: induced subgraph of G which is the complement of a cycle of length > 3.

Theorem (Chudnovsky, Robertson, S., Thomas, 2006)

If G has no odd holes and no odd antiholes then $\chi(G) = \omega(G)$.

Theorem (Tutte, 1948)

There are graphs G with $\omega(G) = 2$ *and* $\chi(G)$ *arbitrarily large.*

Hole: induced subgraph of G which is a cycle of length > 3.
Antihole: induced subgraph of G which is the complement of a cycle of length > 3.

Theorem (Chudnovsky, Robertson, S., Thomas, 2006)

If G has no odd holes and no odd antiholes then $\chi(G) = \omega(G)$.

What happens in between?

Theorem (Tutte, 1948)

There are graphs G with $\omega(G) = 2$ and $\chi(G)$ arbitrarily large.

Hole: induced subgraph of G which is a cycle of length > 3.
Antihole: induced subgraph of G which is the complement of a cycle of length > 3.

Theorem (Chudnovsky, Robertson, S., Thomas, 2006)

If G has no odd holes and no odd antiholes then $\chi(G) = \omega(G)$.

What happens in between?

Conjecture (Gyárfás, 1985)

If G has no odd holes then $\chi(G)$ is bounded by a function of $\omega(G)$.

Theorem (trivial)

If G has no odd holes and $\omega(G) \leq 2$ then $\chi(G) = \omega(G)$.

Theorem (trivial)

If G has no odd holes and $\omega(G) \leq 2$ *then* $\chi(G) = \omega(G)$.

Theorem (Chudnovsky, Robertson, S., Thomas, 2010)

If G has no odd holes and $\omega(G) = 3$ *then* $\chi(G) \leq 4$.

Theorem (trivial)

If G has no odd holes and $\omega(G) \leq 2$ *then* $\chi(G) = \omega(G)$.

Theorem (Chudnovsky, Robertson, S., Thomas, 2010)

If G has no odd holes and $\omega(G) = 3$ *then* $\chi(G) \leq 4$.

Theorem (Scott, S., August 2014)

If G has no odd holes then $\chi(G) \leq 2^{3^{\omega(G)}}$.

Cograph: graph not containing a 4-vertex path as an induced subgraph.

Cograph: graph not containing a 4-vertex path as an induced subgraph.

Lemma

If J is a cograph with $|V(J)| > 1$, then either J or its complement is disconnected.

Cograph: graph not containing a 4-vertex path as an induced subgraph.

Lemma

If J is a cograph with $|V(J)| > 1$, then either J or its complement is disconnected.

Theorem

Let G be a graph, and let $A, B \subseteq V(G)$ be disjoint, where A is stable and $B \neq \emptyset$. Suppose that

- *every vertex in B has a neighbour in A;*
- *there is a cograph J with vertex set A, with the property that for every induced path P with ends in A and interior in B, its ends are adjacent in J if and only if P has odd length.*

Then there is a partition X, Y of B such that every $\omega(G)$-clique in B intersects both X and Y.

The proof

Let G be a graph with no odd hole. We need to show $\chi(G) \leq 2^{3^{\omega(G)}}$.

The proof

Let G be a graph with no odd hole. We need to show $\chi(G) \leq 2^{3^{\omega(G)}}$.

Enough to show:

Assume

- Every graph H with no odd hole and $\omega(H) < \omega(G)$ has $\chi(H) \leq n$
- G has no odd hole.

Then $\chi(G) \leq 48n^3$.

Levelling in G: Sequence $L_0, L_1, L_2, \ldots, L_k$ of disjoint subsets of $V(G)$ where

- $|L_0| = 1$
- each vertex in L_{i+1} has a neighbour in L_i
- for $j > i + 1$ there are no edges between L_i and L_j.

Levelling in G: Sequence $L_0, L_1, L_2, \ldots, L_k$ of disjoint subsets of $V(G)$ where

- $|L_0| = 1$
- each vertex in L_{i+1} has a neighbour in L_i
- for $j > i + 1$ there are no edges between L_i and L_j.

Enough to show:

Assume

- Every graph H with no odd hole and $\omega(H) < \omega(G)$ has $\chi(H) \leq n$
- G has no odd hole
- $L_0, L_1, L_2, \ldots, L_k$ is a levelling in G.

Then $\chi(L_k) \leq 24n^3$.

Parent of $v \in L_{i+1}$ is a vertex in L_i adjacent to v.

L_i has the unique parent property if $i < k$ and every vertex in L_i is the unique parent of some vertex.

Parent of $v \in L_{i+1}$ is a vertex in L_i adjacent to v.

L_i has the unique parent property if $i < k$ and every vertex in L_i is the unique parent of some vertex.

L_i has the parity property if for all $u, v \in L_i$, all induced paths between them with interior in lower levels have the same parity.

Parent of $v \in L_{i+1}$ is a vertex in L_i adjacent to v.

L_i has the unique parent property if $i < k$ and every vertex in L_i is the unique parent of some vertex.

L_i has the parity property if for all $u, v \in L_i$, all induced paths between them with interior in lower levels have the same parity.

Enough to show:

Assume

- Every graph H with no odd hole and $\omega(H) < \omega(G)$ has $\chi(H) \leq n$
- G has no odd hole
- $L_0, L_1, L_2, \ldots, L_k$ is a levelling in G
- $L_0, \ldots, L_{k-1}$ have the parity property
- $L_0, \ldots, L_{k-1}$ have the unique parent property.

Then $\chi(L_k) \leq 24n^3$.

Spine: Path $S = s_0\text{-}s_1\text{-}\cdots\text{-}s_k$ where

- $s_i \in L_i$ for all i
- s_i is the unique parent of s_{i+1} for all $i < k$
- every vertex in $N(S)$ has the same type, and not type 5 or type 6.

Spine: Path $S = s_0\text{-}s_1\text{-}\cdots\text{-}s_k$ where

- $s_i \in L_i$ for all i
- s_i is the unique parent of s_{i+1} for all $i < k$
- every vertex in $N(S)$ has the same type, and not type 5 or type 6.

$N(S)$ is the set of vertices not in S with a neighbour in S.

Spine: Path $S = s_0\text{-}s_1\text{-}\cdots\text{-}s_k$ where

- $s_i \in L_i$ for all i
- s_i is the unique parent of s_{i+1} for all $i < k$
- every vertex in $N(S)$ has the same type, and not type 5 or type 6.

$N(S)$ is the set of vertices not in S with a neighbour in S.

Type of $v \in N(S) \cap L_i$:

Type 1: i even, v adjacent to s_{i-1} and to no other vertex in S

Type 2: i odd, v adjacent to s_{i-1} and to no other vertex in S

Type 3: i even, v adjacent to s_{i-1}, s_i and to no other vertex in S

Type 4: i odd, v adjacent to s_{i-1}, s_i and to no other vertex in S

Type 5: i even, v adjacent to s_i and to no other vertex in S

Type 6: i odd, v adjacent to s_i and to no other vertex in S.

Enough to show:

Assume

- Every graph H with no odd hole and $\omega(H) < \omega(G)$ has $\chi(H) \leq n$
- G has no odd hole
- $L_0, L_1, L_2, \ldots, L_k$ is a levelling in G
- $L_0, \ldots, L_{k-1}$ have the parity property
- $L_0, \ldots, L_{k-1}$ have the unique parent property
- there is a spine.

Then $\chi(L_k) \leq 4n^3$.

L_i satisfies the parent rule if all adjacent $u, v \in L_i$ have the same parents.

L_i satisfies the parent rule if all adjacent $u, v \in L_i$ have the same parents.

Theorem

Suppose

- *G has no odd hole*
- *$L_0, L_1, L_2, \ldots, L_k$ is a levelling in G*
- *$L_0, \ldots, L_{k-1}$ have the parity property*
- *$L_0, \ldots, L_{k-1}$ have the unique parent property*
- *there is a spine.*

Then $L_0, \ldots, L_{k-2}$ satisfy the parent rule.

Enough to show:

Assume

- Every graph H with no odd hole and $\omega(H) < \omega(G)$ has $\chi(H) \leq n$
- G has no odd hole
- $L_0, L_1, L_2, \ldots, L_k$ is a levelling in G
- $L_0, \ldots, L_{k-1}$ have the parity property
- $L_0, \ldots, L_{k-2}$ satisfy the parent rule.

Then $\chi(L_k) \leq 4n^3$.

Enough to show:

Assume

- Every graph H with no odd hole and $\omega(H) < \omega(G)$ has $\chi(H) \leq n$
- G has no odd hole
- $L_0, L_1, L_2, \ldots, L_k$ is a levelling in G
- $L_0, \ldots, L_{k-1}$ have the parity property
- $L_0, \ldots, L_{k-2}$ satisfy the parent rule
- L_{k-2} is stable.

Then $\chi(L_k) \leq 4n^2$.

Enough to show:

Assume

- Every graph H with no odd hole and $\omega(H) < \omega(G)$ has $\chi(H) \leq n$
- G has no odd hole
- $L_0, L_1, L_2, \ldots, L_k$ is a levelling in G
- $L_0, \ldots, L_{k-1}$ have the parity property
- $L_0, \ldots, L_{k-2}$ satisfy the parent rule.
- L_{k-1} is stable.

Then $\chi(L_k) \leq 2n$.

Let $L_0, \ldots, L_t$ be a levelling in G, where L_t is stable and has the parity property.
The graph of jumps on L_t is the graph on L_t, in which u, v are adjacent if all induced paths between u, v with interior in lower levels are odd.

Let $L_0, \dots, L_t$ be a levelling in G, where L_t is stable and has the parity property.
The graph of jumps on L_t is the graph on L_t, in which u, v are adjacent if all induced paths between u, v with interior in lower levels are odd.

Theorem

Suppose that

- *G has no odd hole*
- *$L_0, \dots, L_t$ is a levelling in G*
- *L_t has the parity property*
- *$L_0, \dots, L_{t-1}$ satisfy the parent rule*
- *L_t is stable.*

Then the graph of jumps on L_t is a cograph.

Enough to show:

Assume

- Every graph H with no odd hole and $\omega(H) < \omega(G)$ has $\chi(H) \leq n$
- $L_0, \ldots, L_k$ is a levelling in G
- L_{k-1} has the parity property
- L_{k-1} is stable
- the graph of jumps on L_{k-1} is a cograph.

Then $\chi(L_k) \leq 2n$.

Recall:

Theorem

Let G be a graph, and let $A, B \subseteq V(G)$ be disjoint, where A is stable and $B \neq \emptyset$. Suppose that

- *every vertex in B has a neighbour in A;*
- *there is a cograph J with vertex set A, with the property that for every induced path P with ends in A and interior in B, its ends are adjacent in J if and only if P has odd length.*

Then there is a partition X, Y of B such that every $\omega(G)$-clique in B intersects both X and Y.