Capacity Bounds for Diamond Networks

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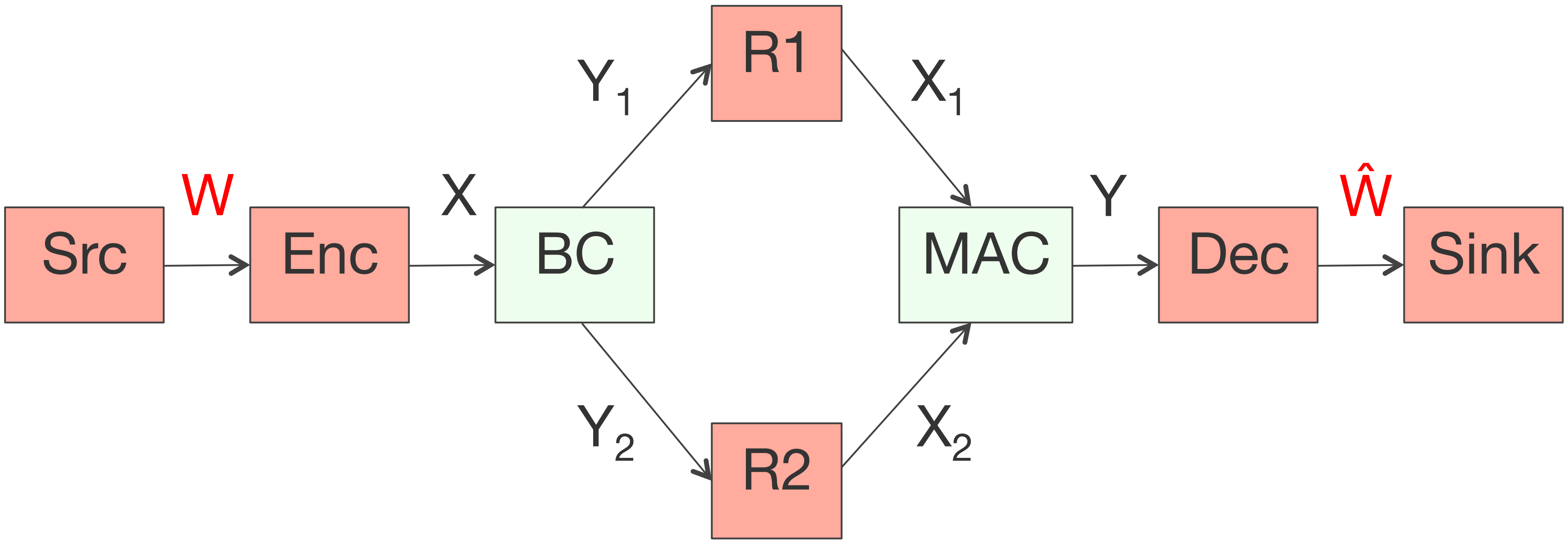
What is a “Diamond Network” ?

- Cascade of a 2-receiver broadcast channel (BC) and a 2-transmitter multi-access channel (MAC)
- Simplifications: (1) MAC is two bit-pipes; (2) BC is two bit-pipes



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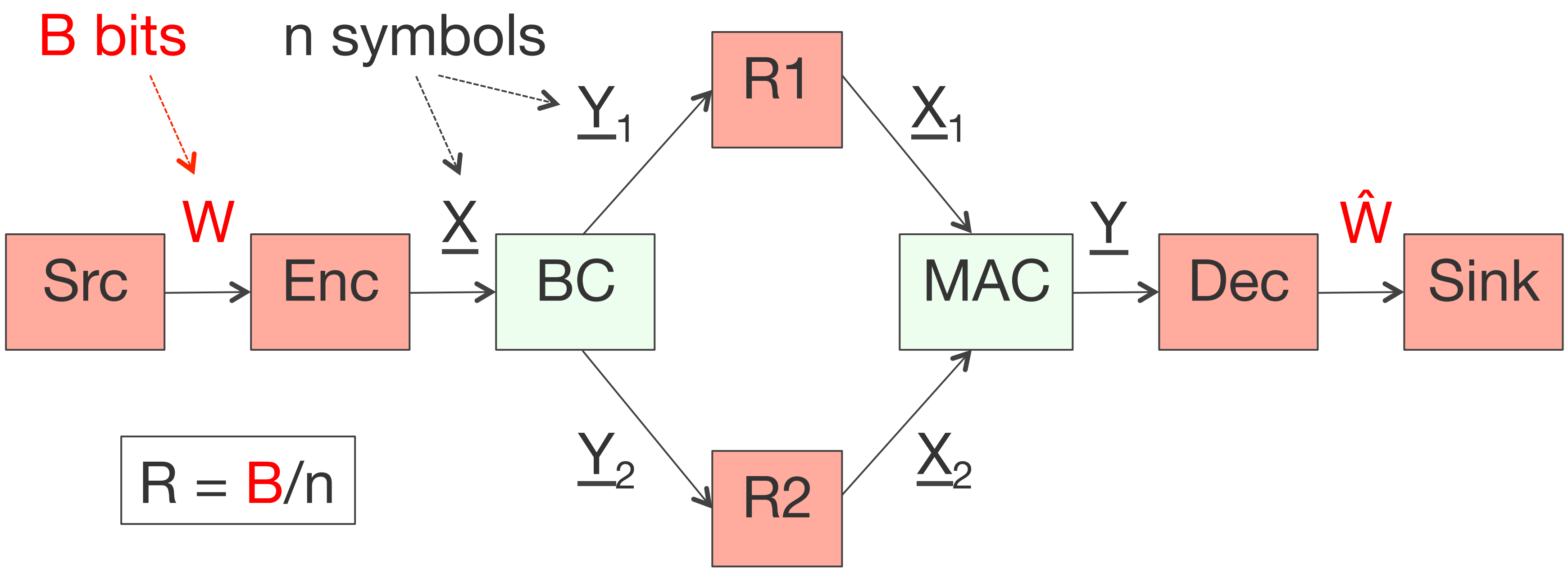


Background

General Problem

- B. E. Schein, Distributed coordination in network information theory. PhD Dissertation, MIT, 2001

MAC is 2 Bit Pipes

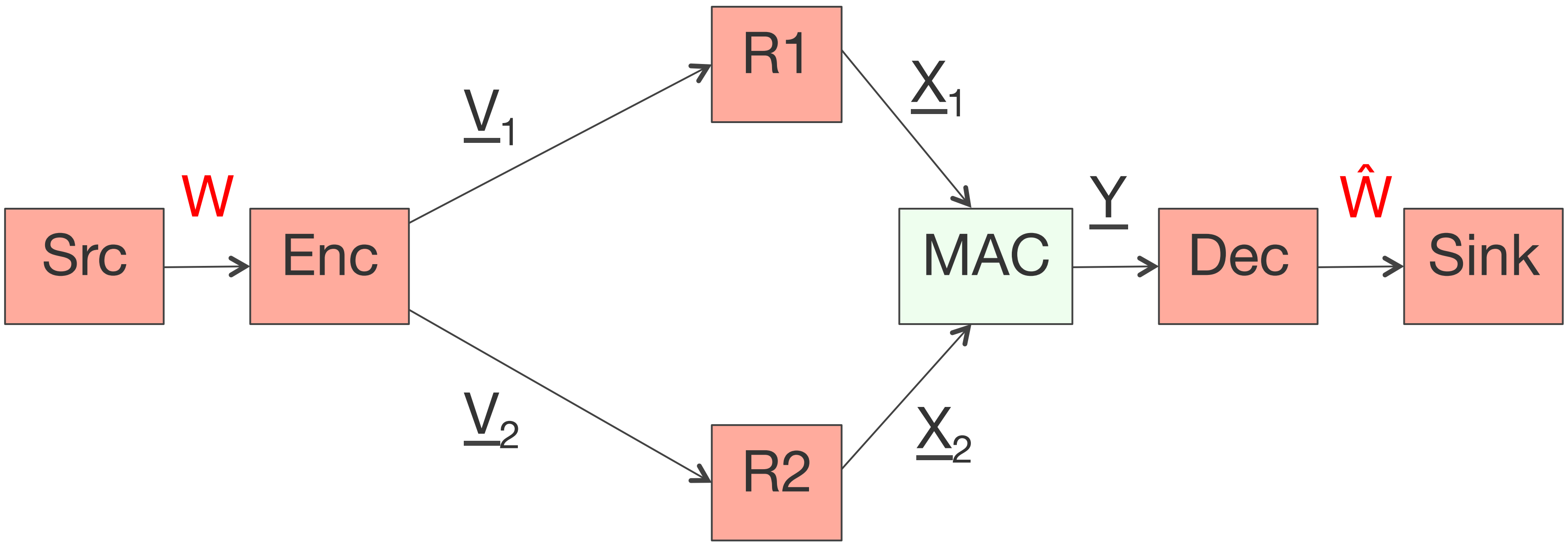
- A. Sanderovich, S. Shamai, Y. Steinberg, G. Kramer, "Communication via decentralized processing," IEEE Trans. IT, 2008

BC is 2 Bit Pipes

- D. Traskov, G. Kramer, "Reliable communication in networks with multi-access interference," ITW 2007
- W. Kang, N. Liu, and W. Chong, "The Gaussian multiple access diamond channel," arxiv 2011 (v1) and 2015 (v2)

Here: BC is two bit pipes

- Capacity limitations C_1 and C_2. Problem seems difficult!
- Gaussian MAC partially solved by Kang-Liu (2011) using Ozarow's trick (1980)
- Contribution: new capacity upper bound for discrete MACs
- Contribution: solved binary adder MAC capacity by extending Mrs. Gerber's Lemma



Outline

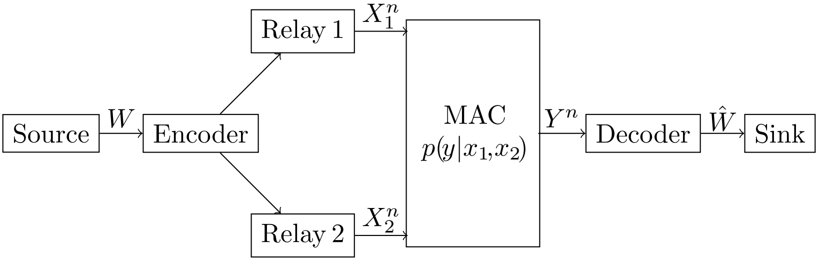
The Problem Setup

A Lower Bound

An Upper-Bound

Examples
- The Gaussian MAC
- The binary adder MAC

The Problem Setup



- W message of rate R

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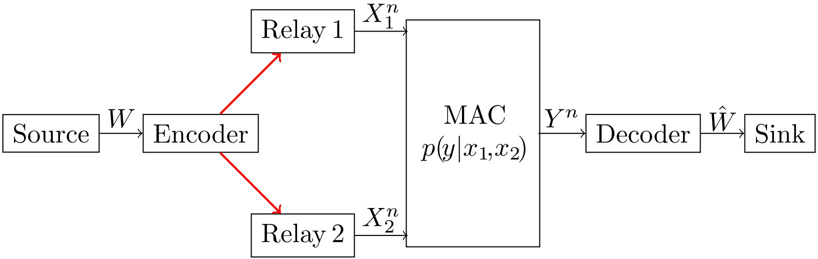

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- Bit-pipes of capacities C_1, C_2

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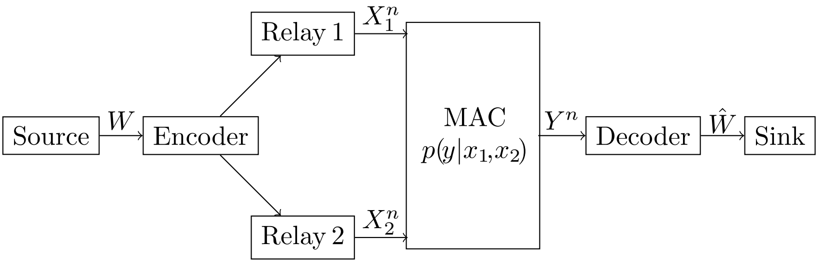
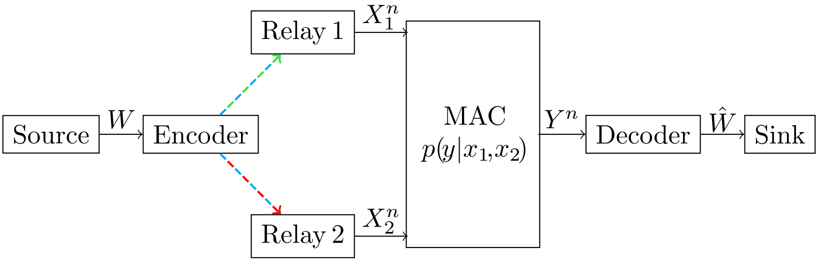


- W message of rate R
- Bit-pipes of capacities C_1, C_2
- Goal: What is the highest rate R such that $\mathbb{P}\mathrm{r}(W \neq \hat{W}) \to 0$?

A Lower Bound



- Rate splitting: $W = (W_{12}, W_1, W_2)$
- Superposition Coding:
  W_{12} encoded in V^n.
  X_1^n, X_2^n superposed on V^n.
- Marton's Coding … a sophisticated superposition

Rate Bounds

- Rate-splitting bounds:

$$
\begin{aligned}
&R_1' + R_2' > I(X_1; X_2|U) \\
&R_{12} + R_1 + R_1' < C_1 \\
&R_{12} + R_2 + R_2' < C_2 \\
&R_{12} + R_1 + R_1' + R_2 + R_2' < I(X_1 X_2; Y) + I(X_1; X_2|U) \\
&R_1 + R_1' + R_2 + R_2' < I(X_1 X_2; Y|U) + I(X_1; X_2|U) \\
&R_2 + R_2' < I(X_2; Y|X_1, U) + I(X_1; X_2|U) \\
&R_1 + R_1' < I(X_1; Y|X_2, U) + I(X_1; X_2|U).
\end{aligned}
$$

- Now apply Fourier-Motzkin elimination

A Lower Bound (Cont.)

Theorem (Lower Bound)

The rate R is achievable if it satisfies the following condition for some pmf $p(v, x_1, x_2, y) = p(v, x_1, x_2)p(y|x_1, x_2)$:

$$R \leq \min \left\{ \begin{array}{l} C_1 + C_2 - I(X_1; X_2|V) \\ C_2 + I(X_1; Y|X_2 V) \\ C_1 + I(X_2; Y|X_1 V) \\ \frac{1}{2}(C_1 + C_2 + I(X_1 X_2; Y|V) - I(X_1; X_2|V)) \\ I(X_1 X_2; Y) \end{array} \right\}$$

$$V \in \mathcal{V}, \ |\mathcal{V}| \leq \min\{|\mathcal{X}_1||\mathcal{X}_2| + 2, |\mathcal{Y}| + 4\}$$

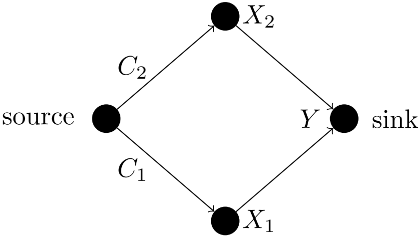
- S. Saeedi Bidokhti, G. Kramer, "Capacity bounds for a class of diamond networks," ISIT 2014
- W. Kang, N. Liu, W. Chong, "The Gaussian multiple access diamond channel," arxiv 1104.3300, v2, 2015

The Cut-Set Bound

Cut-Set bound: R is achievable only if it satisfies the following bounds for some $p(x_1, x_2)$:

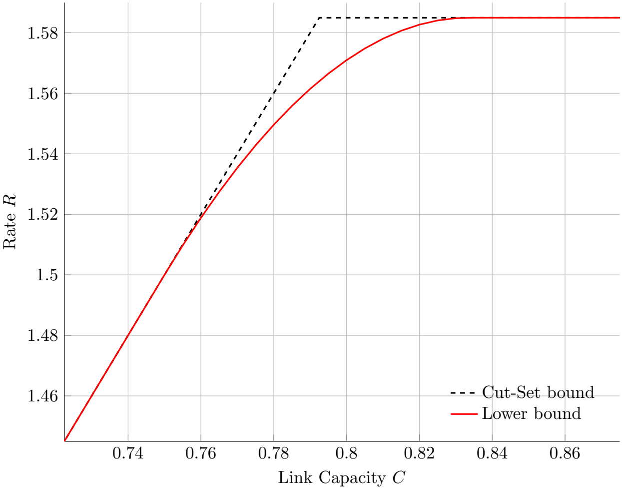
Four Cuts:

$$\begin{aligned} R &\leq C_1 + C_2 \\ R &\leq C_1 + I(X_2; Y|X_1) \\ R &\leq C_2 + I(X_1; Y|X_2) \\ R &\leq I(X_1 X_2; Y). \end{aligned}$$



Example I: binary adder MAC

- $\mathcal{X}_1 = \mathcal{X}_2 = \{0, 1\}, \quad \mathcal{Y} = \{0, 1, 2\}$
- $Y = X_1 + X_2$



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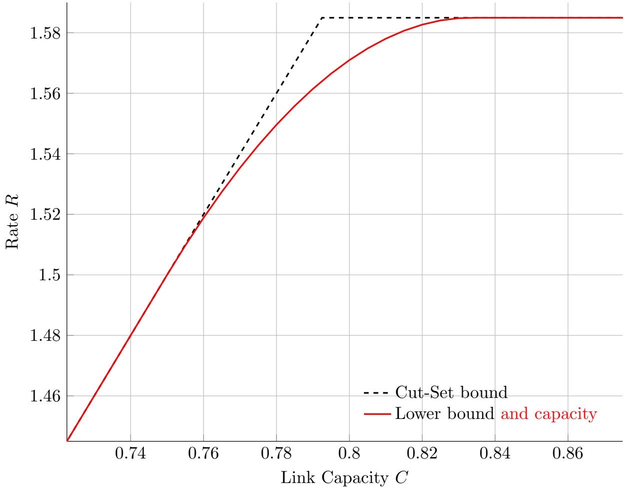
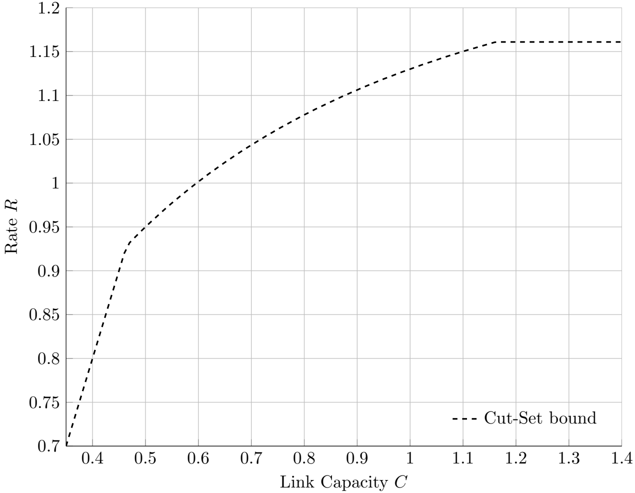


Example II: Gaussian MAC

- $Y = X_1 + X_2 + Z, \quad Z \sim \mathcal{N}(0,1)$
- $\frac{1}{n}\sum_{i=1}^{n} \mathbb{E}(X_{1,i}^2) \leq P_1, \quad \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}(X_{2,i}^2) \leq P_2, \quad P_1 = P_2 = 1$



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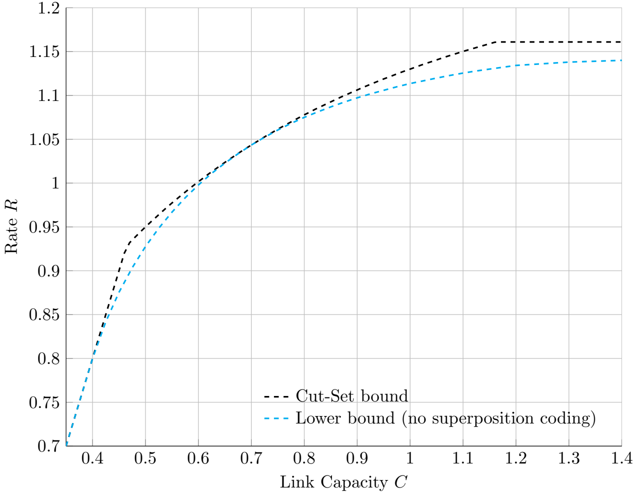

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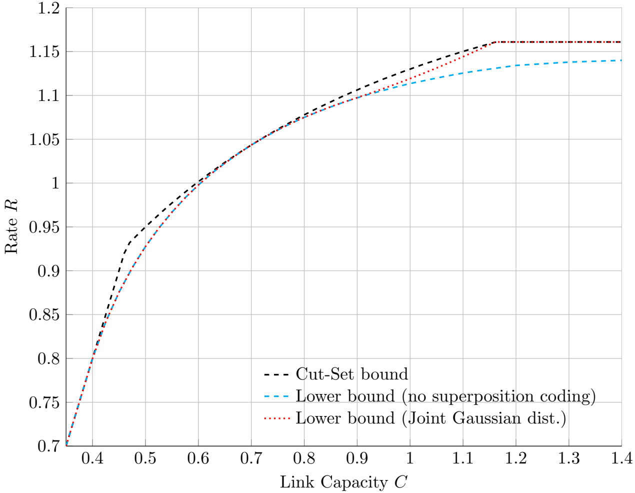

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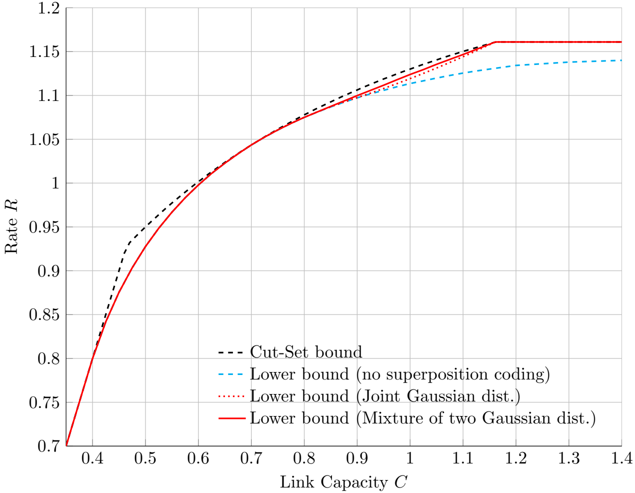

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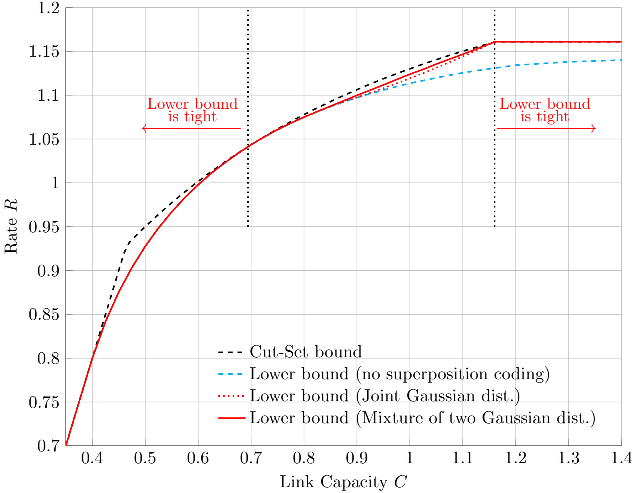
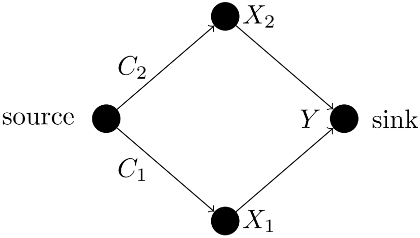


Is the Cut-Set Bound Tight?

Cut-Set bound:

$$\begin{aligned} R &\leq C_1 + C_2 \\ R &\leq C_1 + I(X_2;Y|X_1) \\ R &\leq C_2 + I(X_1;Y|X_2) \\ R &\leq I(X_1X_2;Y). \end{aligned}$$

Maximize over $p(x_1, x_2)$.



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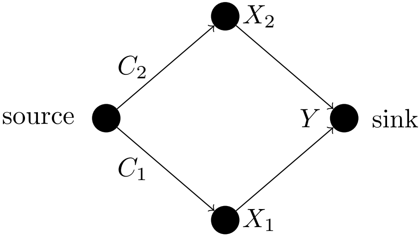

It turns out that the cut-set bound is not tight.

One culprit is the cut ({source}, {R1,R2,sink})

Refining the Cut-Set Bound

- Motivated by [Ozarow'80, KangLiu'11] (cf. [TraskovKramer'07])

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$$nR \leq nC_1 + nC_2 - I(X_1^n; X_2^n)$$

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$$nR \leq nC_1 + nC_2 - I(X_1^n; X_2^n)$$

- For any U^n:

$$\begin{aligned} I(X_1^n; X_2^n) = & \, I(X_1^n X_2^n; U^n) - I(X_1^n; U^n | X_2^n) - I(X_2^n; U^n | X_1^n) \\ & + I(X_1^n; X_2^n | U^n) \end{aligned}$$

Basically the Hekstra-Willems Dependence Balance Bound (IT'89)!
See Gastpar-Kramer (ITW'06)

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$$nR \leq nC_1 + nC_2 - I(X_1^n; X_2^n)$$

- For any U^n:

$$I(X_1^n; X_2^n) \geq I(X_1^n X_2^n; U^n) - I(X_1^n; U^n | X_2^n) - I(X_2^n; U^n | X_1^n)$$

Refining the Cut-Set Bound (Cont.)

$$nR \le nC_1 + nC_2 - I(X_1^n X_2^n; U^n) + I(X_1^n; U^n | X_2^n) + I(X_2^n; U^n | X_1^n)$$

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choose U_i as follows:

$$Y_i \longrightarrow \boxed{p_{U|Y}} \longrightarrow U_i$$

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$$\ldots \leq n\left(C_1 + C_2 + I(X_1 X_2; Y | U) + I(X_1; U | X_2) + I(X_2; U | X_1)\right)$$

New Upper-Bounds (1)

Theorem (Upper Bound I)

The rate R is achievable only if there exists a joint distribution $p(x_1, x_2)$ for which the following inequalities hold for every auxiliary channel $p(u|x_1, x_2, y) = p(u|y)$

$$
\begin{aligned}
R &\leq C_1 + C_2 \\
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- max-min problem

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- max-min problem
- $2R \leq C_1 + C_2 + I(X_1X_2; Y) - I(X_1; X_2) + I(X_1; X_2|U)$

New Upper-Bounds (2)

Theorem (Upper Bound II)

The capacity is bounded from above by

$$\max_{p(x_1,x_2)} \min_{\substack{p(u|x_1,x_2,y)\\ =p(u|y)}} \max_{\substack{p(q|x_1,x_2,y,u)\\ =p(q|x_1,x_2)}} \min\left\{\begin{array}{l} C_1+C_2, \\ C_1+I(X_2;Y|X_1Q), \\ C_2+I(X_1;Y|X_2Q), \\ I(X_1X_2;Y|Q), \\ C_1+C_2-I(X_1;X_2|Q)+I(X_1;X_2|UQ) \end{array}\right\}$$

- $|\mathcal{Q}| \le |\mathcal{X}_1||\mathcal{X}_2| + 3.$

Don't drop the mutual information term
and use Y-to-U channel structure

New Upper-Bounds (2)

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- $|\mathcal{Q}| \le |\mathcal{X}_1||\mathcal{X}_2| + 3$.
- last term is related to the Hekstra-Willems dependence balance bound and can be written as

$$R \le C_1+C_2-I(X_1X_2;U|Q)+I(X_2;U|X_1Q)+I(X_1;U|X_2Q)$$

New Upper-Bounds (2)

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The Gaussian MAC

$Y = X_1 + X_2 + Z$

$Z \sim \mathcal{N}(0, 1)$,
$\frac{1}{n} \sum_{i=1}^{n} \mathbb{E}(X_{1,i}^2) \leq P$,
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Max-Min-Max problem

$$R \leq 2C$$
$$R \leq C + I(X_1; Y | X_2 Q)$$
$$R \leq C + I(X_2; Y | X_1 Q)$$
$$R \leq I(X_1 X_2; Y | Q)$$
$$R \leq \quad C_1 + C_2 - I(X_1 X_2; U | Q) + I(X_1; U | X_2 Q) + I(X_2; U | X_1 Q)$$

The Gaussian MAC

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$$R \leq \quad C_1 + C_2 - I(X_1X_2;U|Q) + I(X_1;U|X_2Q) + I(X_2;U|X_1Q)$$

Choose $U = Y + Z_N$

$Z_N \sim \mathcal{N}(0, N)$

N to be optimized.

The Gaussian MAC

$$Y = X_1 + X_2 + Z$$

$$Z \sim \mathcal{N}(0,1),$$
$$\frac{1}{n}\sum_{i=1}^{n} \mathbb{E}(X_{1,i}^2) \leq P,$$
$$\frac{1}{n}\sum_{i=1}^{n} \mathbb{E}(X_{2,i}^2) \leq P$$

Choose $U = Y + Z_N$

$Z_N \sim \mathcal{N}(0, N)$

N to be optimized.

$$R \leq 2C$$
$$R \leq C + \log\left(1 + P(1-\rho^2)\right)/2$$
$$R \leq C + I(X_2; Y|X_1 Q)$$
$$R \leq I(X_1 X_2; Y|Q)$$
$$R \leq \quad C_1 + C_2 - I(X_1 X_2; U|Q) + I(X_1; U|X_2 Q) + I(X_2; U|X_1 Q)$$

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$$R \leq C + \log\left(1 + P(1-\rho^2)\right)/2$$
$$R \leq C + \log\left(1 + P(1-\rho^2)\right)/2$$
$$R \leq I(X_1X_2;Y|Q)$$
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N to be optimized.

$$R \leq 2C$$
$$R \leq C + \log\left(1 + P(1-\rho^2)\right)/2$$
$$R \leq C + \log\left(1 + P(1-\rho^2)\right)/2$$
$$R \leq \log\left(1 + 2P(1+\rho)\right)/2$$
$$R \leq C_1 + C_2 - I(X_1X_2;U|Q) + I(X_1;U|X_2Q) + I(X_2;U|X_1Q)$$

The Gaussian MAC

$$Y = X_1 + X_2 + Z$$

$$Z \sim \mathcal{N}(0,1),$$
$$\frac{1}{n}\sum_{i=1}^{n} \mathbb{E}(X_{1,i}^2) \leq P,$$
$$\frac{1}{n}\sum_{i=1}^{n} \mathbb{E}(X_{2,i}^2) \leq P$$

$$R \leq 2C$$
$$R \leq C + \log\left(1 + P(1-\rho^2)\right)/2$$
$$R \leq C + \log\left(1 + P(1-\rho^2)\right)/2$$
$$R \leq \log\left(1 + 2P(1+\rho)\right)/2$$

Choose $U = Y + Z_N$

$Z_N \sim \mathcal{N}(0, N)$

N to be optimized.

$$R \leq \quad C_1 + C_2 - I(X_1X_2;U|Q) + \log\left(\frac{1+N+P\left(1-\rho^2\right)}{1+N}\right)$$

The Gaussian MAC (Cont.)

- $U = Y + Z_N,\ Z_N \sim \mathcal{N}(0, N)$

$$
\begin{aligned}
I(X_1X_2;U|Q) &= h(U|Q) - h(U|X_1X_2) \\
&\overset{\text{EPI}}{\geq} \frac{1}{2}\log\left(2\pi eN + 2^{2h(Y|Q)}\right) - \frac{1}{2}\log\left(2\pi e(1+N)\right)
\end{aligned}
$$

$$
I(X_1X_2;Y|Q) = h(Y|Q) - \frac{1}{2}\log\left(2\pi e\right) \geq R
$$

The Gaussian MAC (Cont.)

- $U = Y + Z_N,\ Z_N \sim \mathcal{N}(0, N)$

$$I(X_1X_2;U|Q) = h(U|Q) - h(U|X_1X_2)$$
$$\stackrel{\text{EPI}}{\geq} \frac{1}{2}\log\left(2\pi eN + 2^{2h(Y|Q)}\right) - \frac{1}{2}\log\left(2\pi e(1+N)\right)$$

$$I(X_1X_2;Y|Q) = h(Y|Q) - \frac{1}{2}\log\left(2\pi e\right) \geq R$$

$$R \leq C_1 + C_2 - \frac{1}{2}\log\left(N + 2^{2R}\right) - \frac{1}{2}\log\left(1+N\right)$$
$$+ \log\left(1 + N + P\left(1-\rho^2\right)\right)$$

The Gaussian MAC (Cont.)

- $U = Y + Z_N$, $Z_N \sim \mathcal{N}(0, N)$

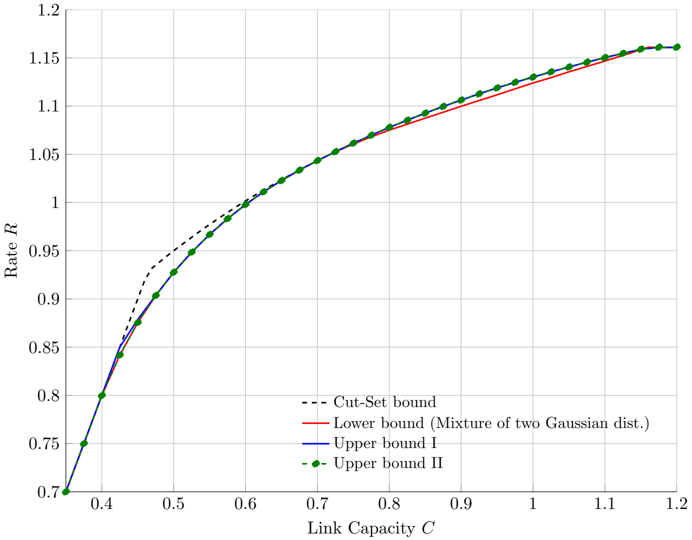
$$I(X_1X_2;U|Q) = h(U|Q) - h(U|X_1X_2)$$
$$\overset{\text{EPI}}{\geq} \frac{1}{2}\log\left(2\pi eN + 2^{2h(Y|Q)}\right) - \frac{1}{2}\log\left(2\pi e(1+N)\right)$$

$$I(X_1X_2;Y|Q) = h(Y|Q) - \frac{1}{2}\log\left(2\pi e\right) \geq R$$

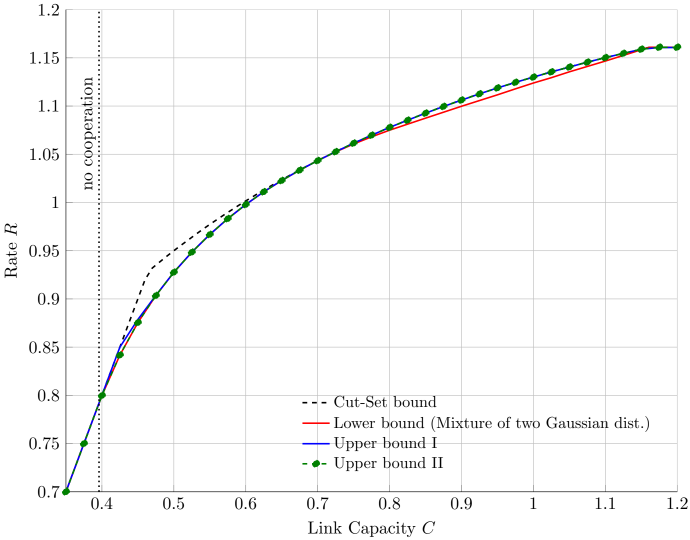
$$R \leq C_1 + C_2 - \frac{1}{2}\log\left(N + 2^{2R}\right) - \frac{1}{2}\log\left(1+N\right) + \log\left(1 + N + P\left(1-\rho^2\right)\right)$$

- Strictly tighter than [KangLiu'11]

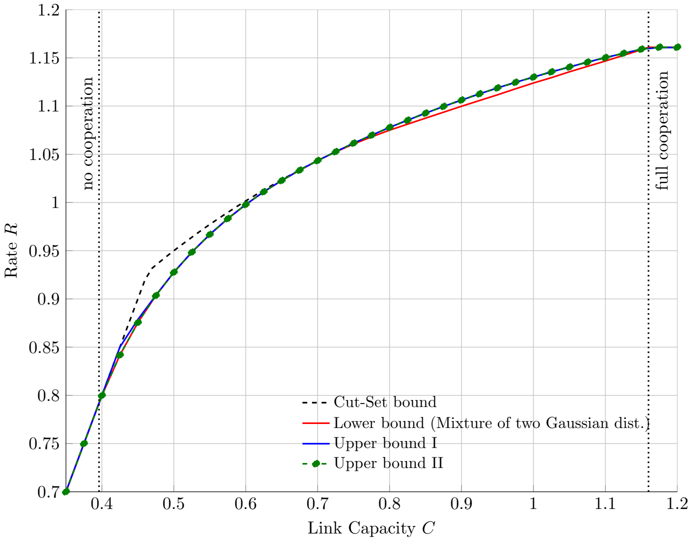
The Gaussian MAC (Cont.)



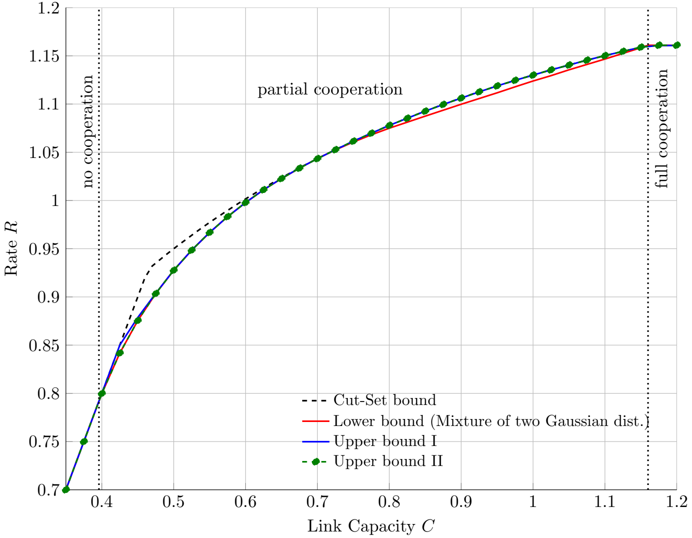
The Gaussian MAC (Cont.)



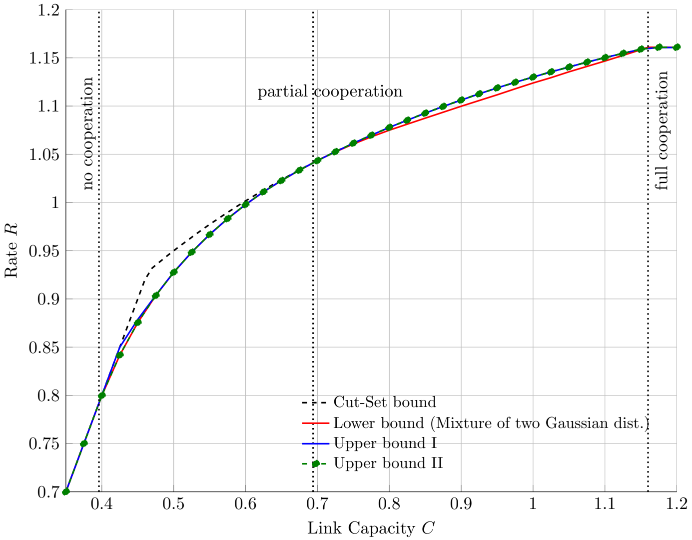
The Gaussian MAC (Cont.)



The Gaussian MAC (Cont.)



The Gaussian MAC (Cont.)



On The Capacity of The Gaussian MAC

Theorem

For a symmetric Gaussian diamond network, the upper bound meets the lower bound for all C such that $C \geq \frac{1}{2}\log(1+4P)$, or

$$C \leq \frac{1}{4}\log\frac{1+2P(1+\rho^{(2)})}{1-\left(\rho^{(2)}\right)^2}$$

where

$$\rho^{(2)} = \sqrt{1+\frac{1}{4P^2}} - \frac{1}{2P}$$

The Optimal Choice of N

- $U = Y + Z_N$ (motivated by [Ozarow'80, KangLiu'11])
- (X_1, X_2) an optimal jointly Gaussian input for the lower bound

$$\begin{bmatrix} P & \lambda^\star P \\ \lambda^\star P & P \end{bmatrix}.$$

- $N = \left(P\left(\frac{1}{\lambda^\star} - \lambda^\star\right) - 1\right)^+$
- $P\left(\frac{1}{\lambda^\star} - \lambda^\star\right) - 1 \geq 0$: $X_1 - U - X_2$ forms a Markov chain– new upper-bound
- $P\left(\frac{1}{\lambda^\star} - \lambda^\star\right) - 1 \leq 0$: the cut-set bound

The Binary Adder MAC

$$Y = X_1 + X_2, \quad \mathcal{X}_1 = \mathcal{X} = \{0,1\}, \quad \mathcal{Y} = \{0,1,2\}$$

$$\begin{aligned}
R &\leq C_1 + C_2 \\
R &\leq C_2 + I(X_1; Y | X_2 Q) \\
R &\leq C_1 + I(X_2; Y | X_1 Q) \\
R &\leq I(X_1 X_2; Y | Q) \\
R &\leq C_1 + C_2 - I(X_1 X_2; U | Q) + I(X_1; U | X_2 Q) + I(X_2; U | X_1 Q)
\end{aligned}$$

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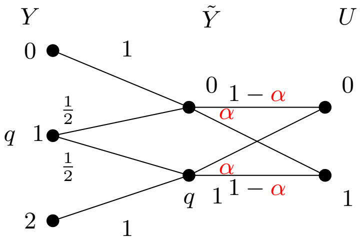
$$Y = X_1 + X_2, \quad \mathcal{X}_1 = \mathcal{X} = \{0, 1\}, \quad \mathcal{Y} = \{0, 1, 2\}$$



$$\begin{aligned}
R &\leq C_1 + C_2 \\
R &\leq C_2 + I(X_1; Y|X_2Q) \\
R &\leq C_1 + I(X_2; Y|X_1Q) \\
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R &\leq C_1 + C_2 - I(X_1X_2; U|Q) + I(X_1; U|X_2Q) + I(X_2; U|X_1Q)
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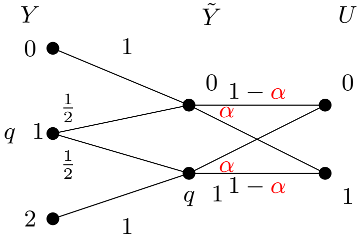


$$\begin{aligned}
R &\leq C_1 + C_2 \\
R &\leq C_2 + h_2(q) \\
R &\leq C_1 + h_2(q) \\
R &\leq 1 + h_2(q) - q \\
R &\leq C_1 + C_2 - I(X_1X_2; U|Q) + 2h_2(\frac{q}{2} \star \alpha) - 2(1-q)h_2(\alpha) - 2q
\end{aligned}$$

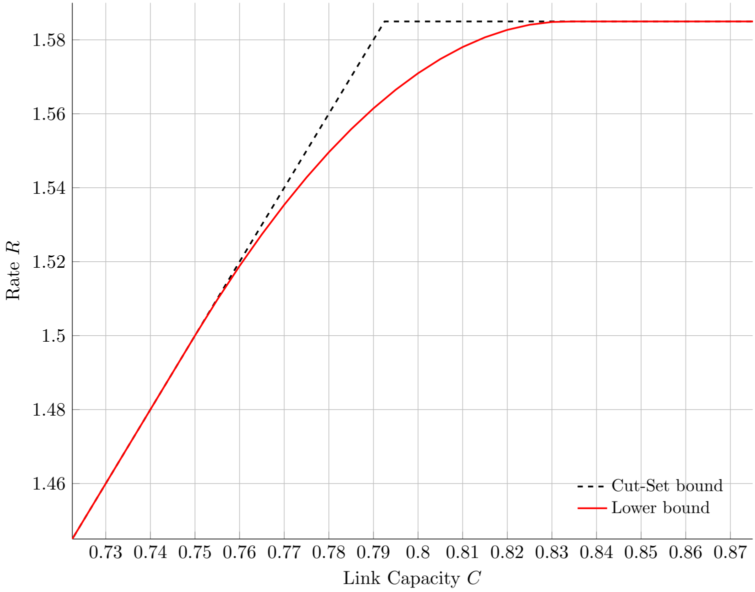
The Interplay in the upper bound

$$I(X_1X_2;U|Q) = H(U|Q) - H(U|X_1X_2)$$

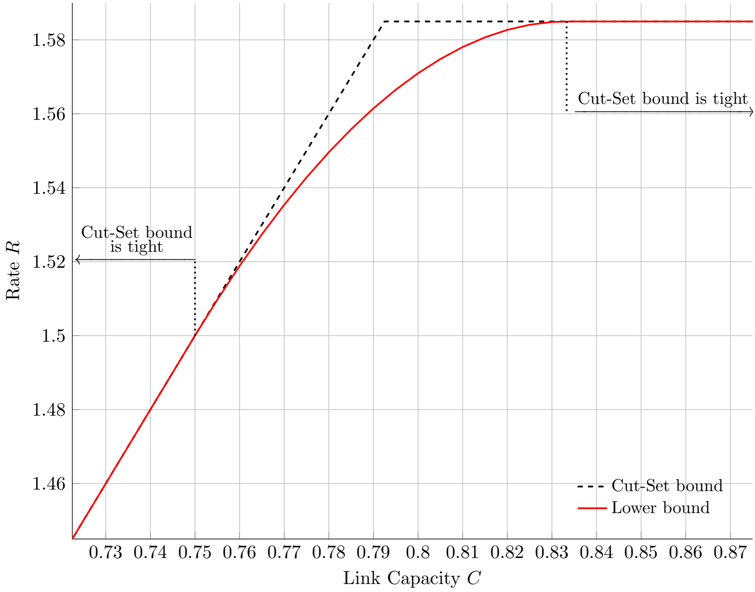
$$\overset{\text{MGL}}{\geq} h_2\left(\alpha \star h_2^{-1}\left(H(\tilde{Y}|Q)\right)\right) - (1-q)h_2(\alpha) - q$$

$$I(X_1X_2;Y|Q) = H(\tilde{Y}|Q) + h_2(q) - q \geq R$$

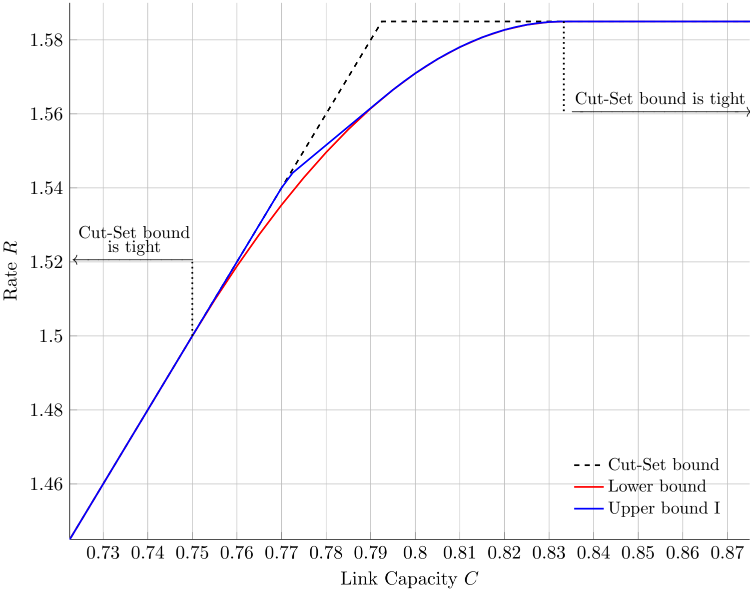
The Binary Adder MAC (Cont.)



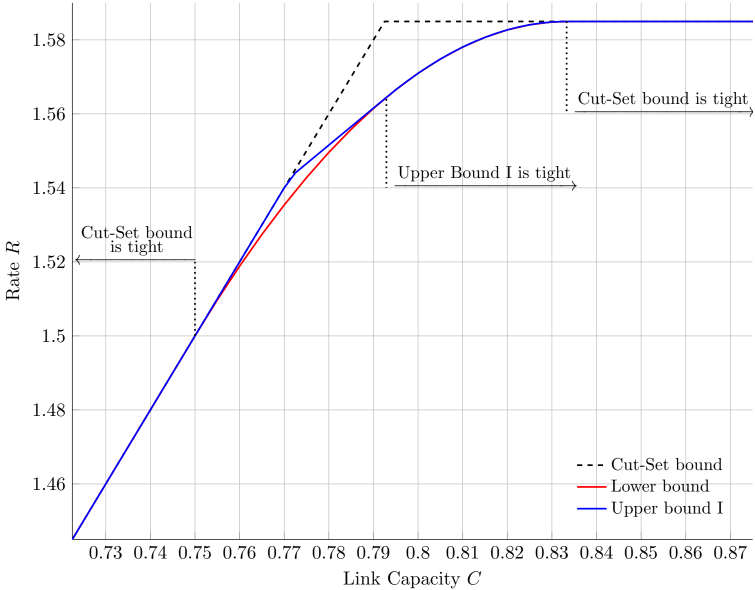
The Binary Adder MAC (Cont.)



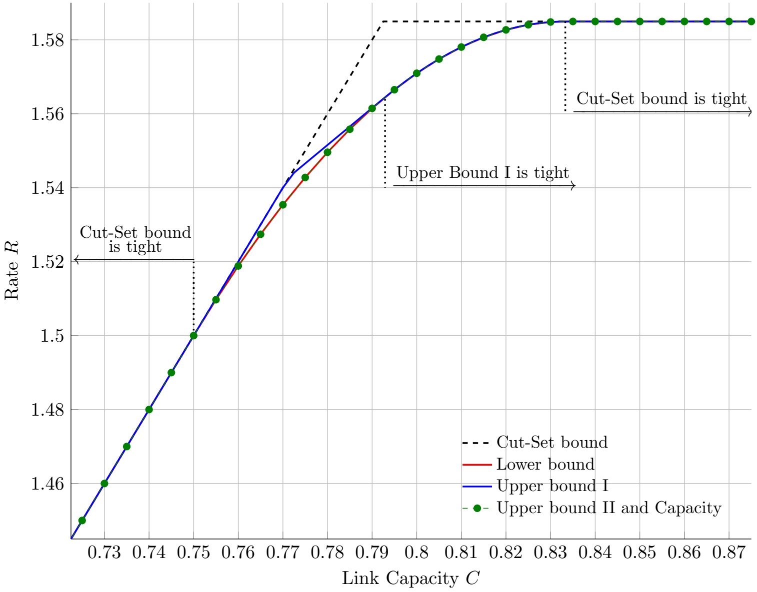
The Binary Adder MAC (Cont.)



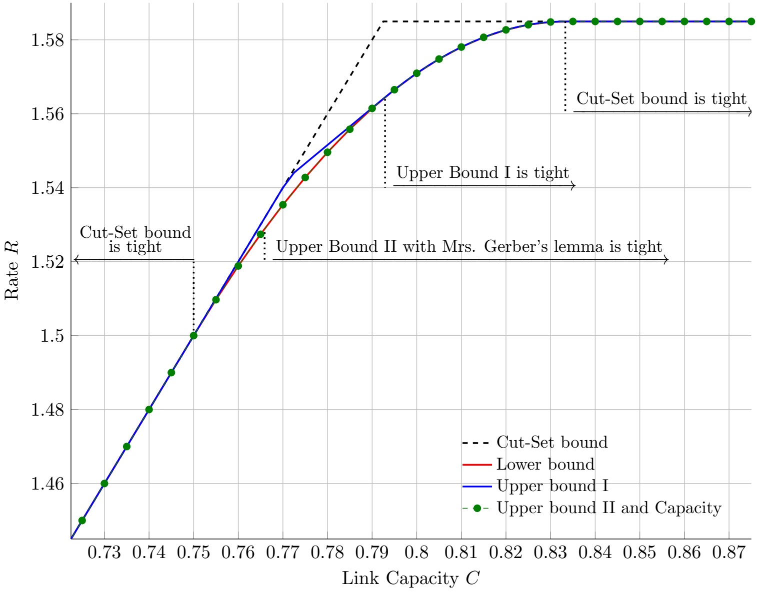
The Binary Adder MAC (Cont.)



The Binary Adder MAC (Cont.)



The Binary Adder MAC (Cont.)



The interplay in the upper bounds

$$R \leq I(X_1X_2;Y|Q)$$

$$R \leq C_1 + C_2 - I(X_1X_2;U|Q) + I(X_2;U|X_1Q) + I(X_1;U|X_2Q)$$

The interplay in the upper bounds

$$
\begin{aligned}
R \leq & I(X_1 X_2; Y|Q) \\
\leq & H(Y|Q) - H(Y|X_1 X_2) \\
R \leq & C_1 + C_2 - I(X_1 X_2; U|Q) + I(X_2; U|X_1 Q) + I(X_1; U|X_2 Q)
\end{aligned}
$$

The interplay in the upper bounds

$$
\begin{aligned}
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\leq & C_1 + C_2 - H(U|Q) - H(U|X_1X_2) + H(U|X_1Q) + H(U|X_2Q)
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$$

The interplay in the upper bounds

$$
\begin{aligned}
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\leq & \; H(Y|Q) - H(Y|X_1X_2) \\
R \leq & C_1 + C_2 - I(X_1X_2;U|Q) + I(X_2;U|X_1Q) + I(X_1;U|X_2Q) \\
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\end{aligned}
$$

- Up to now: Entropy Power Inequality, Mrs. Gerber's Lemma
  1. $\min\{H(U)|H(Y)=t\} \geq f(t)$
  2. $f(t)$ is convex in t

The interplay in the upper bounds

$$
\begin{aligned}
R &\leq I(X_1X_2;Y|Q)\\
&\leq H(Y|Q) - H(Y|X_1X_2)\\
R &\leq C_1 + C_2 - I(X_1X_2;U|Q) + I(X_2;U|X_1Q) + I(X_1;U|X_2Q)\\
&\leq C_1 + C_2 - H(U|Q) - H(U|X_1X_2) + H(U|X_1Q) + H(U|X_2Q)
\end{aligned}
$$

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The interplay in the upper bounds

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\begin{aligned}
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&\le C_1 + C_2 - H(U|Q) - H(U|X_1X_2) + H(U|X_1Q) + H(U|X_2Q)
\end{aligned}
$$

- Up to now: Entropy Power Inequality, Mrs. Gerber's Lemma
  1. $\min\{H(U)|H(Y)=t\} \ge f(t)$
  2. $f(t)$ is convex in t

- What we want to do:
  1. $\min\{H(U) - H(U|X_1) - H(U|X_2)|H(Y)=t\} \ge f(t)$
  2. $f(t)$ is convex in t

The Binary Adder MAC: Upper Bound

$$
\begin{aligned}
R &\le 2C \\
R &\le C + h_2(q) \\
R &\le 1 + h_2(q) - q \\
R &\le 2C - h_2\left(\alpha \star \left(\frac{q}{2} + (1-q)h_2^{-1}\left(\min\left(1, \frac{(R - h_2(q))^+}{1-q}\right)\right)\right)\right) \\
&\quad - (1-q)h_2(\alpha) - q + 2h_2\left(\alpha \star \frac{q}{2}\right)
\end{aligned}
$$

RHS is jointly concave (note signs) in (R,q)

Capacity of The Binary Adder MAC

Theorem

The capacity of diamond networks with binary adder MACs is

$$\max_{0\leq p\leq\frac{1}{2}} \min \begin{cases} C_1 + C_2 - 1 + h_2(p) \\ C_1 + h_2(p) \\ C_2 + h_2(p) \\ h_2(p) + 1 - p. \end{cases}$$

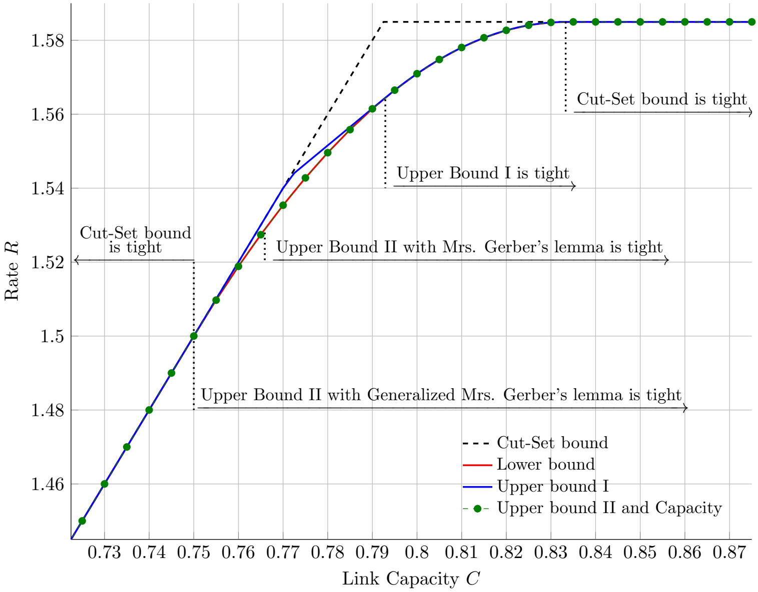
The optimal Choice of α

- Let (X_1, X_2) be an optimizing doubly symmetric binary pmf with parameter $p^\star$ for the lower bound
- α is such that

$$\alpha(1-\alpha) = \left(\frac{p^\star}{2(1-p^\star)}\right)^2$$

and it makes the following Markov chain $X_1 - U - X_2$.

Capacity of The Binary Adder MAC



Summary and Work in Progress

- Lower and Upper bounds on the capacity of a class of diamond networks

- A new upper bound which is in the form of a max-min problem

- Gaussian MACs:
  - improved previous lower and upper bounds
  - characterized the capacity for interesting ranges of bit-pipe capacities.

- Binary adder MAC: fully characterized the capacity

- Work in progress: the general class of 2-relay diamond networks, n-relay diamond networks with orthogonal BC components