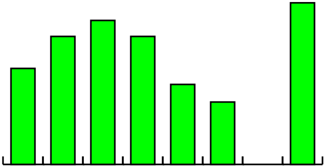
Communication Complexity of Learning Discrete Distributions

Krzysztof Onak
IBM T.J. Watson Research Center

Joint work with **Ilias Diakonikolas**, **Elena Grigorescu**, and **Abhiram Natarajan**.

Distribution Learning and Testing

Input: Stream of independent samples
from an unknown distribution $\mathcal{D}$



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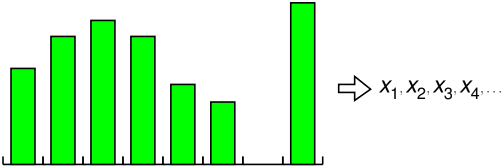

Goal:

Learn the distribution

or test a property

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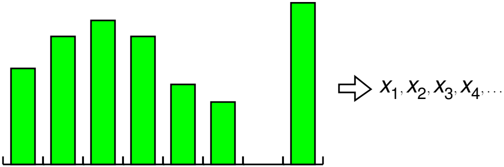

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- Small total variation distance error acceptable
- Traditional focus: sample complexity

Learning Discrete Distributions

$\mathcal{D}$ = probability distribution on $\{1, \ldots, n\}$
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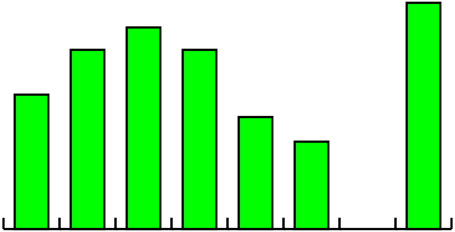

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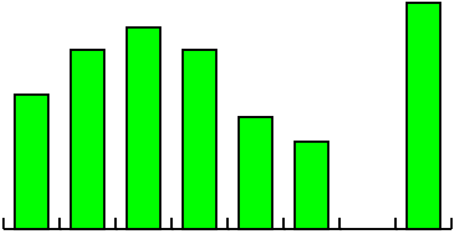

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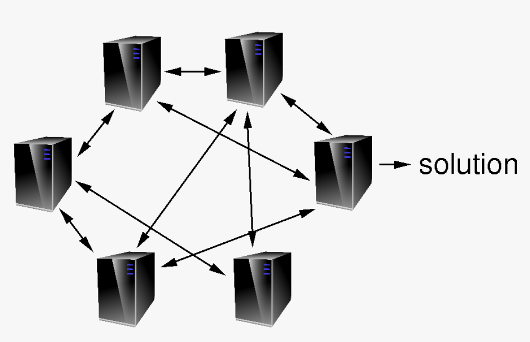
Sample complexity: $\Theta(n/\epsilon^2)$

Communication Complexity

Distributed data: samples held by different players

Example: Samples in different data centers



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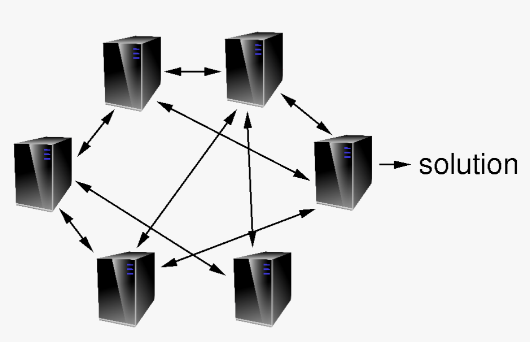

How much do players have to communicate to solve the problem?

Is sublinear communication possible?

"Survey" Complexity

This talk will focus on the simplest setting:

- Each player has one sample
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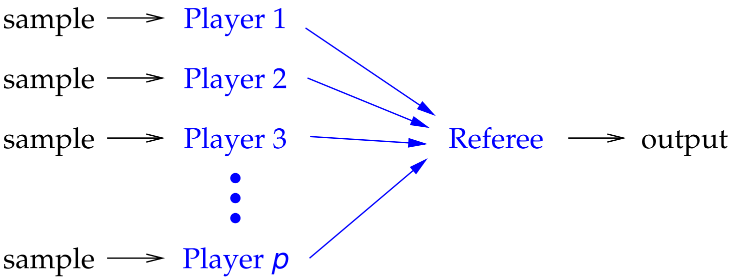
- The referee outputs solution



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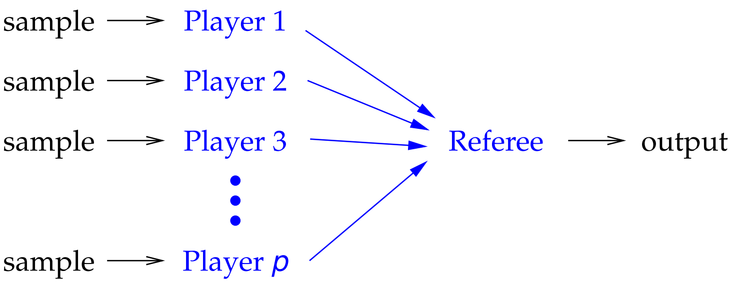

- Each sample is $\Theta(\log n)$ bits
- Can average communication be made $o(\log n)$?

Related Work

A lot of recent interest in communication-efficient learning:

DAW12, ZDW13, ZX15, GMN14, KVW14, LBKW14, SSZ14, DJWZ14, LSLT15, BGMNW15

- Both upper and lower bounds.
- Usually more continuous problems.
- Sample problem: estimating the mean of a Gaussian distribution.

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See Mark Braverman's talk tomorrow

Outline

1. $O(n/\epsilon^2)$ Sample Complexity Review
2. Communication Complexity Lower Bound
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Why this works:

- For every subset of $\{1, \dots, n\}$ the probabilities under $\mathcal{D}$ and $\mathcal{D}'$ within $\epsilon/2$ with probability $1 - 2^{-2n}$
- Union bound: $\|\mathcal{D} - \mathcal{D}'\|_1 \leq \epsilon$ with probability $1 - o(1)$

Lower Bound Review

Fact: Hoeffding's inequality is optimal

- ϵ-biased coin, determine direction of the bias
- $\Omega(\epsilon^{-2})$ coin tosses needed

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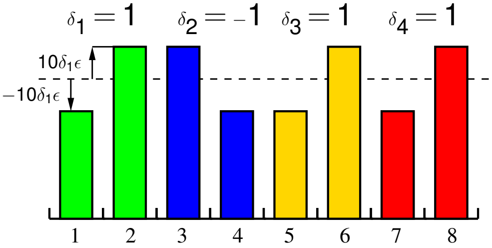

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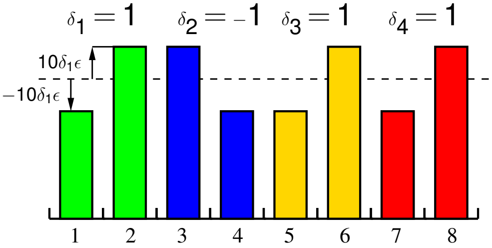

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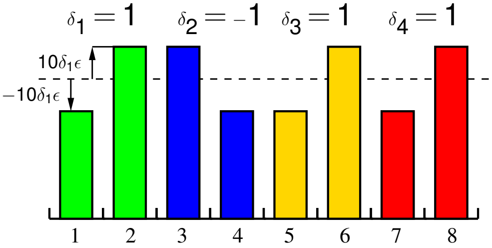

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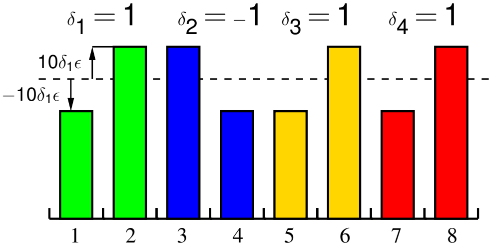

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- This requires $\Omega(n/\epsilon^2)$ samples

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No protocol with $o\left(\frac{n}{\epsilon^2}\log n\right)$
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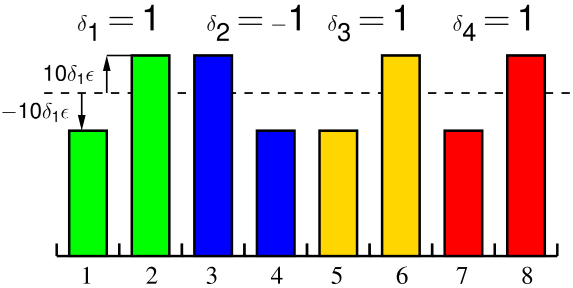
(Can assume at most $O\left(n/\epsilon^2 \log n\right)$ players in the proof)

Hard Distribution

Reuse the hard distribution for sampling:



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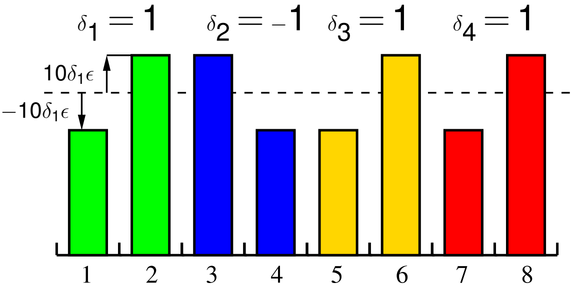

Can assume the protocol is deterministic:

- Slight loss in the probability of success
- Expected communication goes up by constant factor

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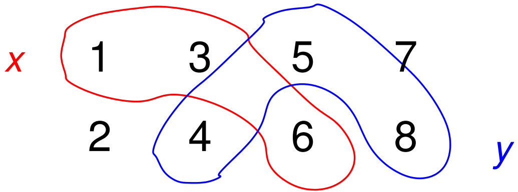
CONTRADICTION!!!

Messages of Single Player

Modify protocol for each pair $2j - 1$ and $2j$:

- Before: x sent for $2j - 1$ and y sent for $2j$
- After: send xy for $2j - 1$ and yx for $2j$



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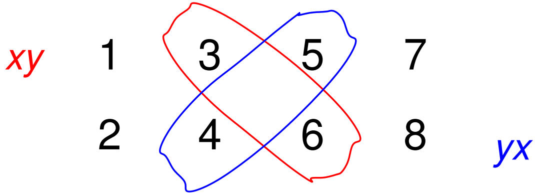

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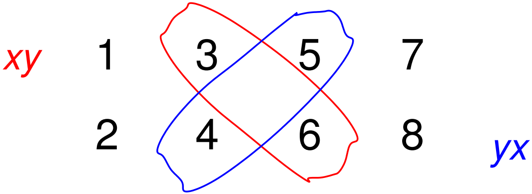

Result:

- Communication complexity only doubles.
- This partitions pairs. Each message reveals bias on a specific subset of pairs.

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1. $|xy| > \frac{\log n}{100}$
   - Happens for $o(n/\epsilon^2)$ fraction of players
   - Can assume the message reveals the sample
   - $I(\text{message}; \delta_i) \leq I(\text{sample}; \delta_i) = O(\epsilon^2/n)$

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2. $|xy| \leq \frac{\log n}{100}$ & $\leq \sqrt{n}$ pairs with these messages
    - Random i: happens with probability $\frac{n^{0.01} \cdot \sqrt{n}}{n}$
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3. $|xy| \leq \frac{\log n}{100}$ & $> \sqrt{n}$ pairs with these messages
    - Can happen always
    - δ_i has little impact on probabilities of xy and yx
    - $I(\text{sample}; \delta_i) = O(\epsilon^2/(n \cdot \#\text{pairs})) = O(\epsilon^2/n^{1.5})$

Total Information about δ_i

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$$\sum_j I(\delta_i; M_j) = o\left(\frac{n}{\epsilon^2}\right) \cdot O\left(\frac{\epsilon^2}{n}\right) + O\left(\frac{n^{0.52}}{\epsilon^2}\right) \cdot O\left(\frac{\epsilon^2}{n}\right)$$
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Algorithm correct with probability $\frac{1}{2} + o(1)$

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Communication complexity bound:

- Assume lengths of all messages $o(\log n)$
- Methods presented here imply:
  - Referee likely learns $n^{-\Omega(1)}$-fraction of samples
  - Other messages provide little information
  - Not enough to distinguish hard instances

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Questions?